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THE GRADIENT OF CERTAIN HARMONIC FUNCTIONS ON MANIFOLDS OF ALMOST NONNEGATIVE RICCI CURVATURE

BY

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ABSTRACT

In this paper we prove that when the Ricci curvature of a Riemannian manifold M^n is almost nonnegative, and a ball $B_L(p) \subset M^n$ is close in Gromov–Hausdorff distance to a Euclidean ball, then the gradient of the harmonic function $\mathbf{b}$ defined in [ChCo1] does not vanish. In particular, these functions can serve as harmonic coordinates on balls sufficiently close to an Euclidean ball. The proof is based on a monotonicity theorem that generalizes monotonicity of the frequency for harmonic functions on $\mathbf{R}^n$.

Introduction

Assume M^n is an n-dimensional Riemannian manifold; its Ricci curvature satisfies

$$\mathrm{Ric}_{M^n} > -(n-1)H. \tag{0.1}$$

Recall a construction of Cheeger-Colding: suppose γ: $[0, L] \to M^n$ is a minimal geodesic; define $b(x)$ by $b(x) = d(x, \gamma(0))$. Assume $\frac{1}{3}L < t < \frac{2}{3}L$. Pick p such that $d(p, \gamma(t)) < \delta L$, and let $\mathbf{b}$: $B_4(p) \to \mathbf{R}$ be the harmonic function with the same boundary value on $\partial B_4(p)$ as b. It was proved in [ChCo1] that

$$|b - \mathbf{b}| < \Psi(\delta, L^{-1}, H|n), \tag{0.2}$$

$$\fint_{B_2(p)} |\nabla \mathbf{b} - \nabla b|^2 < \Psi(\delta, L^{-1}, H|n), \tag{0.3}$$

$$\fint_{B_2(p)} |\operatorname{Hess}_{\mathbf{b}}|^2 < \Psi(\delta, L^{-1}, H|n). \tag{0.4}$$

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Here as in [ChCo1], $\Psi(\delta, L^{-1}, H|n)$ is some nonnegative function of δ, L^{-1}, H, n that goes to 0 when $\delta, L^{-1}, H \to 0$. Also, for any open set W write

$$(0.5) \qquad \fint_W u = \frac{1}{\mathrm{Vol}(W)} \int_W u.$$

Write d_{GH} for the Gromov–Hausdorff distance ([Gr]). The main result of this paper is

THEOREM 0.6: *If L^{-1}, H, δ are sufficiently small, and the Gromov–Hausdorff distance $d_{GH}(B_L(p), B_L(0))$ is sufficiently small ($B_L(0) \subset \mathbf{R}^n$), then $\nabla \mathbf{b} \neq 0$ on $B_1(p)$.*

However, one cannot expect a *uniform* lower bound for $|\nabla \mathbf{b}|$.

Denote by $A(q, r_1, r_2)$ the annulus $\{x | r_1 \leq d(x,q) \leq r_2\}$. For a function u, write $u_{q,r}$ for the average of u on $B_r(q)$. The proof of Theorem 0.6 uses a "three annulus" argument (see [Si], [ChT]); in our case, it is a generalization of the monotonicity of frequency for harmonic functions on $\mathbf{R}^n$ (see [Al], [CoMi1]) to manifolds sufficiently close to $\mathbf{R}^n$:

THEOREM 0.7: *For ϵ small enough (as in Corollary 1.10), there exist δ, $H > 0$ depending only on ϵ such that if a manifold (M,p) satisfies (0.1), $d_{GH}(B_2(p), B_2(0)) < \delta$ ($B_2(0) \subset \mathbf{R}^n$), then for any harmonic function u over $B_2(p)$, and any $q \in B_1(p)$ and $r > 0$ such that $B_r(q) \subset B_{3/2}(p)$, the inequality,*

$$(0.8) \qquad \fint_{A(q,r/2,r)} |u - u_{q,r}|^2 \leq (2^{1+\epsilon})^2 \fint_{A(q,r/4,r/2)} |u - u_{q,r/2}|^2$$

implies

$$(0.9) \qquad \fint_{A(q,r/4,r/2)} |u - u_{q,\frac{r}{2}}|^2 < (2^{1+\frac{\epsilon}{2}})^2 \fint_{A(q,r/8,r/4)} |u - u_{q,r/4}(u)|^2.$$

Theorem 0.7 says that for a harmonic function, sublinear growth on one scale implies sublinear growth on *any* smaller scale.

Recall that the result in (section 6 of) [ChCo1] implies that if M^n is sufficiently close to $\mathbf{R}^n$, then locally, $\mathbf{b}$ is uniformly close to a linear function on $\mathbf{R}^n$. In particular, at some scale (0.8) holds for $\mathbf{b}$. So (0.8) holds for $\mathbf{b}$ on an arbitrarily small scale. This is not possible if $\nabla \mathbf{b} = 0$.

We prove Theorem 0.7 and Theorem 0.6 in Section 2. We argue by contradiction, using a blow up procedure. Assume to the contrary, that (after rescaling) there exists a sequence of manifolds converging to $\mathbf{R}^n$, and a sequence of harmonic functions, $\{u_i\}$, satisfying (0.8), but not (0.9). Here u_i is defined on M_i^n.

It is easy to find a subsequence that converges to some function u_∞ on $\mathbf{R}^n$ (see Definition 1.13). Note the claim that (0.8) is true and (0.9) is not true holds also for u_∞.

It is enough to prove that the function u_∞: $\mathbf{R}^n \to \mathbf{R}$ is also harmonic. Then we get a contradiction to a monotonicity theorem for harmonic functions on $\mathbf{R}^n$; see Corollary 1.10.

By Dirichlet's principle, a harmonic function on B_R is the unique minimizer of the functional $\int_{B_R} |\nabla u|^2$ within the class of functions having the same boundary value on ∂B_R. We prove that u_∞ is energy minimizing by comparing $\int_{B_R(x_\infty)} |\nabla u_\infty|^2$ with $\int_{B_R(x_i)} |\nabla u_i|^2$, which is minimal within the class of functions having the same boundary value on $\partial B_R(x_i) \subset M^n$, $x_i \to x_\infty$.

If we assume $B_1(p) \subset M^n$ is ϵ-close to the Euclidean ball, then as in [ChCo1], [ChCo2], [Co], we have n harmonic functions $\mathbf{b}_1$ $\mathbf{b}_2$, ..., $\mathbf{b}_n$. By a straightforward ODE argument, Theorem 0.6 implies:

COROLLARY 0.10: *Assume M^n satisfies (0.1), $B_1(p) \subset M^n$ is ϵ-close to the Euclidean ball. If ϵ is small enough, then there is a $\psi(\epsilon, H) > 0$ such that there is an open subset U with $B_\psi(p) \subset U$, on which $\mathbf{b_i}$ is a harmonic coordinate system. In particular, U is diffeomorphic to a Euclidean ball.*

This can be viewed as an analogue of the Reifenberg type theorem of Cheeger-Colding; see the appendix of [ChCo2].

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1. Some analysis

Assume u is harmonic on $B_2(0) \subset \mathbf{R}^n$. It is well known that u is analytic and each homogeneous part of u is also harmonic. So we can write

$$u(x) = \sum_{i=0}^{\infty} a_i r^i s_i. \tag{1.1}$$

Here $r = |x|$, $\{s_i\}$ is an orthonormal basis of eigenfunctions of the intrinsic Laplacian on $\partial B_1(0)$, $-\Delta_{\partial B_1(0)} s_i = i(n+i-2)s_i$. Put

$$I(r) = \frac{1}{\mathrm{Vol}(\partial B_r(0))} \int_{\partial B_r(0)} u^2. \tag{1.2}$$

Then

$$I(r) = \sum_{i=0}^{\infty} a_i^2 r^{2i}. \tag{1.3}$$

The so-called "monotonicity of the frequency" theorem on $\mathbf{R}^n$ implies that $I(r)/I(r/2)$ is a nondecreasing function of r; see [Al], [CoMi1]. Here we need the following variant:

LEMMA 1.4: *For $\epsilon > 0$ sufficiently small, if u is harmonic, then*

$$I(r) \leq (2^{1+\epsilon})^2 I(r/2) \tag{1.5}$$

implies

$$I(r/2) < (2^{1+\epsilon/3})^2 I(r/4). \tag{1.6}$$

Proof: By (1.3), (1.5) is equivalent to

$$\sum_{i=2}^{\infty} a_i^2 r^{2i}\Big(1 - \frac{2^{2\epsilon}}{4^{i-1}}\Big) \leq a_0^2(2^{2+2\epsilon} - 1) + a_1^2 r^2 (2^{2\epsilon} - 1). \tag{1.7}$$

On the other hand, (1.6) is equivalent to

$$\sum_{i=2}^{\infty} a_i^2 r^{2i} \frac{1}{4^i}\Big(1 - \frac{2^{2\epsilon/3}}{4^{i-1}}\Big) < a_0^2(2^{2+2\epsilon/3} - 1) + a_1^2 r^2 \frac{1}{4}(2^{2\epsilon/3} - 1). \tag{1.8}$$

Thus, it suffices to show, for $i \geq 2$,

$$\frac{1}{4^i}\Big(1 - \frac{2^{2\epsilon/3}}{4^{i-1}}\Big)\Big/\Big(1 - \frac{2^{2\epsilon}}{4^{i-1}}\Big) < \min\Big(\frac{2^{2+2\epsilon/3} - 1}{2^{2+2\epsilon} - 1}, \frac{2^{2\epsilon/3} - 1}{4(2^{2\epsilon} - 1)}\Big). \tag{1.9}$$

The left side is less than $1/16$, and when $\epsilon \to 0$, the limit of the right hand side is $1/12$. So (1.5) implies (1.6) for ϵ small. ∎

COROLLARY 1.10: *Assume u is harmonic. If*

$$\fint_{A(0,r/2,r)} u^2 \leq (2^{1+\epsilon})^2 \fint_{A(0,r/4,r/2)} u^2, \tag{1.11}$$

then

$$\fint_{A(0,r/4,r/2)} u^2 < (2^{1+\epsilon/3})^2 \fint_{A(0,r/8,r/4)} u^2. \tag{1.12}$$

Next, suppose $(M_i^n, \mathrm{Vol}_i) \xrightarrow{d_{GH}} (M_\infty, \mu_\infty)$ in the measured Gromov–Hausdorff sense, i.e., the sequence $\{M_i^n\}$ converges in the Gromov–Hausdorff sense to M_∞, and for any $x_i \to x_\infty$ ($x_i \in M_i$) and $R > 0$, $\mathrm{Vol}_i(B_R(x_i)) \to \mu_\infty(B_R(x_\infty))$. For any sequences of manifolds with Ricci curvature bounded from below, after possible renormalization of the measures, one can find a subsequence that converges in the measured Gromov–Hausdorff sense; see [ChCo2]. When M_∞ is an n-manifold, $M_i^n \xrightarrow{d_{GH}} M_\infty$ in the Gromov–Hausdorff sense implies $M_i^n \xrightarrow{d_{GH}} M_\infty$ in the measured Gromov–Hausdorff sense; see [Co]. We refer to [Ch], [ChCo2], [GLP] for general background.

Definition 1.13: Suppose $K_i \subset M_i^n \xrightarrow{d_{GH}} K_\infty \subset M_\infty$, and $\{f_i\}$ is a sequence of equicontinuous functions; f_i: $K_i \to \mathbf{R}$, $i = 1, 2, \ldots, \infty$. Assume Φ_i: $K_\infty \to K_i$ are ϵ_i-Gromov–Hausdorff approximations, $\epsilon_i \to 0$. If $f_i \circ \Phi_i$ converge to f_∞ uniformly, we say that $f_i \to f_\infty$ **uniformly** over $K_i \xrightarrow{d_{GH}} K_\infty$.

For simplicity, in the above context we also say $f_i \to f_\infty$ uniformly on K. When we write $f_i(x) \to f_\infty(x)$, we actually mean that $f_i \to f_\infty$ uniformly and there is a sequence $x_i \in M_i^n$, $x_i \to x_\infty \in M_\infty$ such that $f_i(x_i) \to f_\infty(x_\infty)$.

We point out the Arzela–Ascoli theorem can be generalized to the case in which the functions live on different spaces, namely, when $M_i \xrightarrow{d_{GH}} M_\infty$, for any bounded, equicontinuous sequence $\{f_i\}$ (here f_i is a function on M_i), there is a subsequence that converges uniformly to some continuous function f_∞ on M_∞. The proof is straightforward.

Definition 1.14: We say $f_i \to f_\infty$ in the L^2 sense if, for all $\epsilon > 0$, there is a decomposition $f_i = \phi_i + \eta_i$, where $\phi_i \to \phi_\infty$ uniformly and $\limsup |\eta_i|_{L^2} \leq \epsilon$, $|\eta_\infty|_{L^2} \leq \epsilon$.

The following is an extension of the Rellich–Kondrakov theorem:

LEMMA 1.15: *Assume* $\mathrm{Ric}_{M_i} \geq -(n-1)H$, $B_1(p_i) \subset M_i^n \xrightarrow{d_{GH}} B_1(p_\infty) \subset M_\infty$ *in the measured Gromov–Hausdorff sense;* f_i *is a function on* M_i^n ($i = 1, 2, \ldots$). *Assume*

$$\int_{B_1(p_i)} f_i^2 \leq N_1, \tag{1.16}$$

$$\int_{B_1(p_i)} |\nabla f_i|^2 \leq N_2. \tag{1.17}$$

Then there is a subsequence that converges in L^2 *over any given converging sequence of compact subsets* $k_i \xrightarrow{d_{GH}} K_\infty$.

Proof: The argument is similar to those in [CoMi2] and [Ch].

Fix a scale $r > 0$.

For each $i = 1, 2, \ldots, \infty$, we partition $B_1(p_i)$ into M disjoint subsets $S_{1,i}$, $S_{2,i}, \ldots, S_{M,i}$ such that $B_r(x_{j,i}) \subset S_{j,i} \subset B_{2r}(x_{j,i})$, $S_{j,i} \xrightarrow{d_{GH}} S_{j,\infty}$, $j = 1, \ldots, M$. Assume $(f_i)_{x_{j,i},2r}$ is the average of f_i on $B_{2r}(x_{j,i})$. First, by (1.16) and relative volume comparison,

$$((f_i)_{x_{j,i},2r})^2 \le \fint_{B_{2r}(x_{j,i})} f_i^2 \le C_1 r^{-n} N_1. \tag{1.18}$$

Since $\mathrm{Ric}_{M_i} \ge -(n-1)H$ there is a type $(2,2)$ Poincare inequality. So we have

$$\begin{aligned} \int_{S^1_{s,i}} |f_i - (f_i)_{x_{s,i},2r}|^2 &\le \int_{B_{2r}(x_{s,i})} |f_i - (f_i)_{x_{s,i},2r}|^2 \\ &\le r^2 \tau(n,H) \int_{B_{2r}(x_{s,i})} |\nabla f_i|^2. \end{aligned} \tag{1.19}$$

So by (1.17), f_i can be uniformly approximated in L^2 by functions that are constant on each set $S_{j,i}$. In view of the estimates (1.18), (1.19), the lemma follows from a standard diagonal argument. ∎

Assume f_i is a Lipschitz function on M_i^n, $(i = 1, 2, \ldots, \infty)$. Define

$$E(f_i) = \int_{B_R(p_i)} |\nabla f_i|^2 \quad (i = 1, 2, \ldots, \infty). \tag{1.20}$$

Here we only need the case that the limit space is the Euclidean space; see [Di], compare [Ch], [ChCo4].

Lemma 1.21 (Lower semicontinuity of energy): *Suppose f_i are C^2 functions over M_i^n, $\Delta f_i = 0$, $f_i \to f_\infty$ uniformly over the sequence of converging balls $B_{2R}(p_i) \subset M_i^n \xrightarrow{d_{GH}} B_{2R}(p_\infty) \subset \mathbf{R}^n$, and there is a uniform gradient estimate for f_i:*

$$|\nabla f_i| < G. \tag{1.22}$$

Then we have

$$E(f_\infty) \le \liminf_{i \to \infty} E(f_i). \tag{1.23}$$

Proof: As in Section 6 of [ChCo1], we can get an integral bound for the Hessian of f_i on the ball $B_1(p_i)$: recall the Bochner formula

$$\frac{1}{2}\Delta(|\nabla f_i|^2) = |\mathrm{Hess}_{f_i}|^2 + < \nabla \Delta f_i, \nabla f_i > + \mathrm{Ric}(\nabla f_i, \nabla f_i). \tag{1.24}$$

Multiply by a cut-off function ϕ with $\operatorname{supp}\phi \subset B_r \subset B_1(q_i)$, $\phi|_{B_{r/2}} = 1$, $|\nabla\phi| \leq c(n,r)$, $|\Delta\phi| \leq c(n,r)$; see Theorem 6.33 of [ChCo1]. Since f_i is harmonic,

$$\frac{1}{2}\phi\Delta(|\nabla f_i|^2) = \phi|\mathrm{Hess}_{f_i}|^2 + \phi\,\mathrm{Ric}(\nabla f_i, \nabla f_i). \tag{1.25}$$

Integrate by parts,

$$\int_{B_r}\frac{1}{2}(|\nabla f_i|^2)\Delta\phi = \int_{B_r}\phi|\mathrm{Hess}_{f_i}|^2 + \int_{B_r}\phi\,\mathrm{Ric}(\nabla f_i, \nabla f_i). \tag{1.26}$$

By the assumption on Ricci curvature, and note that by construction $|\Delta\phi|$ is uniformly bounded, we have a uniform upper bound for $\int_{B_r}\phi|\mathrm{Hess}_{f_i}|^2$.

So by Lemma 1.15 we can assume that some subsequence of $|\nabla f_i|$ converges to a function Γ on $B_R(p_\infty) \subset M_\infty$ in L^2. Assume $x_\infty \in \mathbf{R}^n$, and there is some subset $A(x_\infty) \subset \mathbf{R}^n$ such that Γ is continuous on $A(x_\infty)$, $x_\infty \in A(x_\infty)$ is a density point of $A(x_\infty)$. By Luzin's theorem, these properties hold for almost all x_∞. For such x_∞, we prove

$$|\nabla f_\infty(x_\infty)| \leq \Gamma(x_\infty). \tag{1.27}$$

Clearly, (1.27) implies our lemma.

To prove (1.27), it is enough to prove, for all $\psi > 0$, if $l = d(x_\infty, y)$ is sufficiently small, then

$$|f_\infty(x_\infty) - f_\infty(y)| \leq d(y, x_\infty)(\Gamma(x_\infty) + 6\psi). \tag{1.28}$$

By the gradient estimate of f_i (so of f_∞), if (1.28) is not true for some y_0, then for all $y \in B_{l\psi/L}(y_0)$,

$$|f_\infty(x_\infty) - f_\infty(y)| > d(y, x_\infty)(\Gamma(x_\infty) + 5\psi). \tag{1.29}$$

Pick $x_i, y_{0,i} \in M_i^n$, $x_i \to x_\infty$, $y_{0,i} \to y_0$, $d(x_i, y_{0,i}) = l$. Then for i big enough, for all $y_i \in B_{l\psi/L}(y_{0,i})$ and all minimal geodesic γ_i connecting x_i and y_i,

$$\int_{\gamma_i}|\nabla f_i| \geq d(x_i, y_i)(\Gamma(x_\infty) + 4\psi). \tag{1.30}$$

Since $|\nabla f_i|$ is uniformly bounded by L, a simple computation shows that along every γ_i we must have

$$|\nabla f_i| > \Gamma(x_\infty) + 2\psi, \tag{1.31}$$

on a subset of γ_i which has length at least $2\psi l/(L-\Gamma(x_\infty))$. Put

$$T_i = \{v \in T_{x_i} | v = \gamma'(0) \text{ for some minimal geodesic } \gamma \text{ from } x_i \text{ to } y_i \in B_{l\psi/L}(y_{0,i})\}; \tag{1.32}$$

we must have $H^{n-1}(T_i) > C(n, L, \psi) H^{n-1}(\partial B_1(0))$, where H^{n-1} is the surface area over $\partial B_1(0)$ in the tangent space T_{x_i}. Combine this with (1.31), if l is small enough, by the proof of the Bishop–Gromov inequality, for sufficiently big i,

$$\frac{\mathrm{Vol}(\{z_i \in B_l(x_i) | |\nabla f_i(z_i)| > \Gamma(x_\infty) + 2\psi\})}{\mathrm{Vol}(B_l(x_i))} \geq C(x_\infty, n, L, \psi) > 0. \tag{1.33}$$

Now $|\nabla f_i|$ converge to Γ in L^2, so (1.33) is also true for $i = \infty$. We get a contradiction to the choice of x_∞. ∎

We also recall one form of the transplantation theorem of Cheeger (see Lemma 10.7 of [Ch]):

LEMMA 1.34: *Assume $M_i \xrightarrow{d_{GH}} \mathbf{R}^n$; f_∞ is a Lipschitz function on $B_R(x_\infty) \subset \mathbf{R}^n$, $x_i \to x_\infty$. Then there is a sequence of Lipschitz functions $\{f_i\}$ that converges uniformly to f_∞; here f_i is defined on $B_R(x_i) \subset M_i$. Moreover, one can require that*

$$\limsup_{i\to\infty} \mathbf{Lip} f_i \leq \mathbf{Lip} f_\infty, \tag{1.35}$$

$$\limsup_{i\to\infty} |\nabla f_i|_{L^2} \leq |\nabla f_\infty|_{L^2}. \tag{1.36}$$

We use subscript i and write f_i, p_i, etc. to denote function, point, etc. on M_i^n.

2. Proof of Theorem 0.7

Assume Theorem 0.7 is not true. Then there is a sequence of manifolds (M_i^n, p_i, q_i) together with a sequence of harmonic functions u_i^* and

$$B_{r_i}(q_i) \subset B_{3/2}(p_i) \subset B_2(p_i) \subset M_i^n$$

such that (0.8) holds while (0.9) is not true, $B_2(p_i) \xrightarrow{d_{GH}} B_2(0) \subset \mathbf{R}^n$, also the lower bound in Ricci curvature goes to 0 as $i \to \infty$.

We rescale the ball $B_{r_i}(q_i)$ to a ball of radius 1; for simplicity, from now on we just denote it by $B_1(q_i)$ (we can assume $r_i < 1$, so the lower bounds in Ricci curvature are improved). We have $B_2(q_i) \xrightarrow{d_{GH}} B_2(0) \subset \mathbf{R}^n$ in the measured Gromov–Hausdorff sense; see Theorems 0.1 and 0.8 in Colding's paper [Co]. So

$$⨍_{A(q_i,\frac{1}{2},1)} |u_i^* - (u_i^*)_{q_i,1}|^2 \leq (2^{1+\epsilon})^2 ⨍_{A(q_i,\frac{1}{4},\frac{1}{2})} |u_i^* - (u_i^*)_{q_i,\frac{1}{2}}|^2, \tag{2.1}$$

while by our assumption

$$(2.2)\qquad \fint_{A(q_i,\frac14,\frac12)} |u_i^* - (u_i^*)_{q_i,\frac12}|^2 \geq (2^{1+\epsilon/2})^2 \fint_{A(q_i,\frac18,\frac14)} |u_i^* - (u_i^*)_{q_i,\frac14}|^2.$$

Define

$$u_i = \frac{u_i^* - (u_i^*)_{q_i,\frac12}}{|\,(u_i^* - (u_i^*)_{q_i,\frac12})|\,_{L^2(B_1(q_i))}}.$$

So u_i still satisfies (2.1), (2.2). Moreover,

$$(2.4)\qquad \int_{B_{\frac12}(q_i)} u_i = 0, \quad \int_{B_1(q_i)} |u_i|^2 = 1.$$

The following results are standard:

LEMMA 2.5: *Assume M^n is a smooth manifold,* $\mathrm{Ric}_M > -(n-1)H$. *If u is harmonic over $B_{R+d}(p)$, then*

$$\sup_{B_R(p)} |u| \leq C(n,H,d/R)|\,u|\,_{L^2(B_{R+d}(p))}.$$

The proof of Lemma 2.5 can be found in [Li] (one can also use Moser iteration).

We can now apply the Cheng–Yau gradient estimate. For $r < R$, on $B_r(p)$ we have

$$(2.7)\qquad |\nabla u| \leq G(r/R, d/R, H, n)|\,u|\,_{L^2(B_{R+d}(p))}.$$

So for any $\theta > 0$, u_i are uniformly bounded over $B_{1-\theta}(q_i) \subset M_i^n$, equicontinuous over $B_{1-2\theta}(q_i) \subset M_i^n$. Thus, without loss of generality, we can assume that the sequence $\{u_i\}$ converges uniformly on compact sets to a locally Lipschitz function u_∞ on the open ball $B_1(0) \subset \mathbf{R}^n$.

LEMMA 2.8: *u_∞ is harmonic.*

Proof: Assume, on the contrary, that u_∞ is not harmonic over a ball

$$B_\lambda(p^*) \subset\subset B_1(0).$$

By solving the Dirichlet problem on $B_\lambda(p^*)$ we can find v_∞ with the same boundary value as u_∞ over ∂B_λ, but with smaller energy, say

$$(2.9)\qquad \int_{B_\lambda(p^*)} |\nabla v_\infty|^2 < \int_{B_\lambda(p^*)} |\nabla u_\infty|^2 - 2\Psi.$$

By obvious density properties, we can change v_∞ slightly so that v_∞ agrees with u_∞ on a neighborhood of $\partial B_\lambda(p^*)$. By Lemma 1.21, for i big enough,

$$(2.10)\qquad \int_{B_\lambda(p^*)} |\nabla v_\infty|^2 \le \int_{B_{\lambda,i}(p^*)} |\nabla u_i|^2 - \Psi.$$

So by (the proof of) Lemma 1.34 (see Section 10 of [Ch]), for i big enough we can find a function v_i with the same boundary value on ∂B_i as u_i but with smaller energy. That contradicts the fact that u_i is harmonic. ∎

Proof of Theorem 0.7: Assume the theorem is not true. As in the discussion at the beginning of this section, after rescaling, we get a sequence of harmonic functions $\{u_i\}$ satisfying (2.1), (2.2) and (2.4) and they converge uniformly over compact subsets to a function u_∞ defined on $B_1(0) \subset \mathbf{R}^n$.

By (2.1), (2.2), (2.4), $u_\infty \neq 0$. By (2.4) and the L^2 convergence, the average value of u_∞ over $B_{\frac{1}{2}}(0)$ is 0. By Lemma 2.8, u_∞ is harmonic, so the averages $(u_\infty)_{0,1}$, $(u_\infty)_{0,\frac{1}{2}}$, and $(u_\infty)_{0,\frac{1}{4}}$ are all zero. In particular, u_∞ is not a constant.

Clearly, (2.1), (2.2) are true for u_∞ over the open ball $B_1(0) \subset \mathbf{R}^n$. This easily leads to a contradiction to Corollary 1.10. ∎

Proof of Theorem 0.6: For any q, assume r is the biggest number such that $A(q,r/2,r) \subset B_{3/2}(p)$. Then as in Section 4 of [ChCo1], we have

$$(2.11)\qquad \fint_{A(q,r/2,r)} |\mathbf{b} - \mathbf{b}_{q,r}|^2 \le (2^{1+\epsilon})^2 \fint_{A(q,r/4,r/2)} |\mathbf{b} - \mathbf{b}_{q,r/2}|^2,$$

if H, δ are small enough, L is big enough, $d_{GH}(B_L(p), B_L(0))$ small enough; see the Introduction. Clearly, we can apply Theorem 0.7 by induction to get

$$(2.12)\qquad \fint_{A(q,r/2^{k+1},r/2^k)} |\mathbf{b} - \mathbf{b}_{q,r/2^k}|^2 \ge (2^{1+\epsilon})^{-2k} \fint_{A(q,r/2,r)} |\mathbf{b} - \mathbf{b}_{q,r}|^2.$$

Assume that $\nabla \mathbf{b}(q) = 0$. Then for all $0 < \beta < 1$ there is a number $C > 0$ such that

$$(2.13)\qquad \fint_{A(q,r/2^{k+1},r/2^k)} |\mathbf{b} - \mathbf{b}_{q,r/2^k}|^2 \le C((r/2^k)^{1+\beta})^2,$$

for k big enough. In particular, we can pick $\beta > \epsilon$. So for k big enough,

$$(2.14)\qquad Cr^{2+2\delta} \ge 2^{2k(\beta-\epsilon)} \fint_{A(q,r/2,r)} |\mathbf{b} - \mathbf{b}_{q,r}|^2.$$

This implies $\mathbf{b}$ must be constant; we get a contradiction (compare (0.2)). ∎

References

[Al] F. Almgren, *Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents*, in *Minimal Submanifolds and Geodesics* (Proceedings of a Japan–United States Seminar, Tokyo, 1977), North-Holland, Amsterdam–New York, 1979, pp. 1–6.

[Ch] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geometric and Functional Analysis **9** (1999), 428–517.

[ChCo1] J. Cheeger and T. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Annals of Mathematics (2) **144** (1996), 189–237.

[ChCo2] J. Cheeger and T. Colding, *On the structure of spaces with Ricci curvature bounded below, Part I*, Journal of Differential Geometry **46** (1997), 406–480.

[ChCo4] J. Cheeger and T. Colding, *On the structure of spaces with Ricci curvature bounded below, Part 3*, Journal of Differential Geometry, to appear.

[ChT] J. Cheeger and G. Tian, *On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay*, Inventiones Mathematicae **118** (1994), 493–571.

[Co] T. Colding, *Ricci curvature and volume convergence*, Annals of Mathematics (2) **145** (1997), 477–501.

[CoMi1] T. Colding and W. Minicozzi II, *Harmonic functions with polynomial growth*, Journal of Differential Geometry **46** (1997), 1–77.

[CoMi2] T. Colding and W. Minicozzi II, *Harmonic functions on manifolds*, Annals of Mathematics (2) **146** (1997), 725–747.

[Di] Y. Ding, *Heat kernels and Green's functions on limit spaces*, Communications in Analysis and Geometry, to appear.

[Gr] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Progress in Mathematics, 152, Birkhäuser Boston, Inc., Boston, MA, 1999.

[Li] P. Li, *Lecture Notes on Geometric Analysis*, Lecture Notes Series, 6, Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, Seoul, 1993.

[SY] R. Schoen and S-T. Yau, *Lectures on Differential Geometry*, International Press, Cambridge, MA, 1994.

[Si] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Annals of Mathematics (2) **118** (1983), 525–571.